New linear independence measures for values of q-hypergeometric series

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1 Introduction

Let $q = q_1/q_2 \in \mathbb{Q}$, where $q_1, q_2 \in \mathbb{Z} \setminus \{0\}$, $\gcd(q_1, q_2) = 1$, $|q_1| > |q_2|$. Put

$$\gamma = \frac{\log |q_2|}{\log |q_1|}. \tag{1.1}$$

Let $P(z) \in \mathbb{Q}[z]$ with $d := \deg P \geqslant 1$. Assume that $P(q^n) \neq 0$ for all $n \in \mathbb{Z}_{>0}$. Consider the function

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{k=1}^{n} P(q^k)}.$$

In this note we prove the following theorem.

Theorem 1. *Let $\alpha_1, \ldots, \alpha_m \in \mathbb{Q}^*$ be such that the following conditions hold:*

1. $\alpha_j \alpha_k^{-1} \notin q^{\mathbb{Z}}$ *for all* $j \neq k$,
2. $\alpha_j \notin P(0) q^{\mathbb{Z}_{>0}}$ *for all* j.

Let $s_1, \ldots, s_m \in \mathbb{Z}_{>0}$. Put

$$S = s_1 + \ldots + s_m, \tag{1.2}$$

$$M = \begin{cases} dS + 1/2 + \sqrt{d^2 S^2 + 1/4}, & P(z) = p_d z^d,\ p_d \in \mathbb{Q}^*, \\ dS + 1 + \sqrt{dS(dS+1)} & \textit{otherwise}. \end{cases} \tag{1.3}$$

Suppose that

$$\gamma < \frac{1}{M},$$

where γ is given by (1.1); then the numbers

$$1, f^{(\sigma)}(\alpha_j q^k) \qquad (1 \leqslant j \leqslant m, 0 \leqslant k < d, 0 \leqslant \sigma < s_j)$$

are linearly independent over $\mathbb{Q}$. Moreover, there exists a positive constant $C_0 = C_0(q, P, m, \alpha_j, s_j)$ such that for any vector $\vec{A} = (A_0, A_{j,k,\sigma}) \in \mathbb{Z}^{1+dS} \setminus \{\vec{0}\}$ we have

$$\left| A_0 + \sum_{j=1}^{m} \sum_{k=0}^{d-1} \sum_{\sigma=0}^{s_j - 1} A_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) \right| \geqslant H^{-\mu - C_0/\sqrt{\log H}},$$

where $H = \max\{\max_{j,k,\sigma} |A_{j,k,\sigma}|, 2\}$,

$$\mu = \frac{M-1}{1 - M\gamma}. \tag{1.4}$$

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The case when all roots of P are rational and $P(0)=0$ was proved in [2] with a larger value for the quantity (1.3) if $P(z)\neq p_d z^d$ (see also [4]). The qualitative part of the general case for $q\in\mathbb{Z}$ was essentially proved in [1], where it was assumed that $\alpha_j\notin P(0)q^{\mathbb{Z}}$ for all j.

Recently the author [3] proved quantitative results in the general case under a milder condition posed on q but with the estimate of the form $\exp\bigl(-C(\log H)^{3/2}\bigr)$, $C=\text{const}$. We modify the method of [3] to prove Theorem 1.

2 Construction of auxiliary linear forms

Fix $\alpha_1,\dots,\alpha_m\in\mathbb{C}^*$, $s_1,\dots,s_m\in\mathbb{Z}_{>0}$. By $\vec{x}$ denote the vector of variables $\vec{x}=(x_0,x_{j,k,\sigma})$, where $1\leqslant j\leqslant m$, $0\leqslant k<d$, $0\leqslant\sigma<s_j$.

Consider the sequences of linear forms

$$u_n=u_n(\vec{x})=\sum_{j=1}^{m}\sum_{k=0}^{d-1}\sum_{\sigma=0}^{s_j-1}\sigma!\binom{n}{\sigma}\bigl(\alpha_j q^k\bigr)^{n-\sigma}x_{j,k,\sigma}\in\mathbb{C}[\vec{x}]\qquad(n\in\mathbb{Z}),\tag{2.1}$$

$$\begin{aligned}v_n=v_n(\vec{x})&=\prod_{k=1}^{n}P(q^k)\cdot\left(x_0+\sum_{l=0}^{n}\frac{u_l(\vec{x})}{\prod_{k=1}^{l}P(q^k)}\right)=\\&=x_0\prod_{k=1}^{n}P(q^k)+\sum_{l=0}^{n}u_l(\vec{x})\prod_{k=l+1}^{n}P(q^k)\in\mathbb{C}[\vec{x}]\qquad(n\in\mathbb{Z}_{\geqslant0}).\end{aligned}\tag{2.2}$$

It's readily seen that

$$v_n=P(q^n)v_{n-1}+u_n\qquad(n\geqslant1)\tag{2.3}$$

with $v_0=x_0+u_0=x_0+\sum_{j=1}^{m}\sum_{k=0}^{d-1}\sum_{\sigma=0}^{s_j-1}x_{j,k,\sigma}$.

Further, let $\mathcal{B}$ be the backward shift operator given by

$$\mathcal{B}\bigl(\xi(n)\bigr)=\xi(n-1).$$

For $a\in\mathbb{C}$ introduce the difference operator

$$\mathcal{D}_a=\mathcal{I}-a\mathcal{B},\tag{2.4}$$

where $\mathcal{I}$ is the identity operator, $\mathcal{I}\bigl(\xi(n)\bigr)=\xi(n)$. Note that these operators commute with each other. For example, we have

$$\mathcal{B}\left(\mathcal{D}_a\bigl(\xi(n)\bigr)\right)=\mathcal{D}_a\bigl(\xi(n-1)\bigr).$$

It's well known that for $a\in\mathbb{C}^*$ and $p(z)\in\mathbb{C}[z]$ with $\deg p\leqslant t\in\mathbb{Z}_{\geqslant0}$ we have

$$\mathcal{D}_a^{t+1}\bigl(p(n)a^n\bigr)=0\qquad(n\in\mathbb{Z}).\tag{2.5}$$

Also, it is readily seen that for $a,b\in\mathbb{C}$ with $b\neq0$ we have

$$\mathcal{D}_a\bigl(b^n\xi(n)\bigr)=b^n\mathcal{D}_{ab^{-1}}\bigl(\xi(n)\bigr).\tag{2.6}$$

Further, for $l,n\in\mathbb{Z}_{\geqslant0}$ with $n\geqslant Sl$, where S is given by (1.2), put

$$v_{l,n}=v_{l,n}(\vec{x})=\prod_{k=1}^{l}\prod_{j=1}^{m}\mathcal{D}_{\alpha_j q^{-k}}^{s_j}\bigl(v_n(\vec{x})\bigr):=\left(\prod_{k=1}^{l}\prod_{j=1}^{m}\mathcal{D}_{\alpha_j q^{-k}}^{s_j}\right)\bigl(v_n(\vec{x})\bigr)\in\mathbb{C}[\vec{x}].\tag{2.7}$$

Finally, let

$$\varepsilon_0=\begin{cases}1, & P(z)=p_d z^d,\ p_d\in\mathbb{Q}^*,\\0 & \text{otherwise.}\end{cases}\tag{2.8}$$

Lemma 1. *Let $l \geqslant d$, $\vec{\omega} = (\omega_0, \omega_{j,k,\sigma}) \in \mathbb{C}^{1+dS}$. Assume that for $0 \leqslant \nu < l$ and $n \geqslant S\nu$ we have*

$$|v_{\nu,n}(\vec{\omega})| \leqslant |q|^{-\nu n+(S-\varepsilon_0/d)\nu^2/2+an+b},$$

where $a > 0$ and b don't depend on ν and n. Then for $n \geqslant Sl$ we have

$$|v_{l,n}(\vec{\omega})| \leqslant |q|^{-ln+(S-\varepsilon_0/d)l^2/2+an+b+a+c'},$$

where c' is a positive constant depending only on q, P, m, α_j, s_j.

Proof. Since $l \geqslant d$, it follows from (2.1) and (2.5) that

$$\prod_{k=1}^{l}\prod_{j=1}^{m}\mathcal{D}^{s_j}_{\alpha_j q^{d-k}}\big(u_n(\vec{\omega})\big) = 0 \qquad (n \in \mathbb{Z}).$$

Therefore, from (2.3) we have

$$\prod_{k=1}^{l}\prod_{j=1}^{m}\mathcal{D}^{s_j}_{\alpha_j q^{d-k}}\big(v_{n+1}(\vec{\omega}) - P(q^{n+1})v_n(\vec{\omega})\big) = 0 \qquad (n \geqslant Sl). \tag{2.9}$$

Let

$$P(z) = \sum_{\nu=0}^{d} p_\nu z^\nu.$$

Then in view of (2.6) the relation (2.9) can be rewritten in the form

$$p_d v_{l,n}(\vec{\omega}) = q^{-d(n+1)}\prod_{k=1}^{l}\prod_{j=1}^{m}\mathcal{D}^{s_j}_{\alpha_j q^{d-k}}\big(v_{n+1}(\vec{\omega})\big) - \sum_{\nu=1}^{d} p_{d-\nu}q^{-\nu(n+1)}\prod_{k=1}^{l}\prod_{j=1}^{m}\mathcal{D}^{s_j}_{\alpha_j q^{\nu-k}}\big(v_n(\vec{\omega})\big). \tag{2.10}$$

It follows from the conditions of the lemma and (2.6) that for $1 \leqslant \nu \leqslant d$ we have

$$\begin{aligned}
\left|q^{-\nu n}\prod_{k=1}^{l}\prod_{j=1}^{m}\mathcal{D}^{s_j}_{\alpha_j q^{\nu-k}}\big(v_{n+\varepsilon_0}(\vec{\omega})\big)\right| &= \left|q^{-\nu n}\prod_{k=0}^{\nu-1}\prod_{j=1}^{m}\mathcal{D}^{s_j}_{\alpha_j q^{k}}\big(v_{l-\nu,n+\varepsilon_0}(\vec{\omega})\big)\right| \leqslant \\
&\leqslant |q|^{-\nu n}\prod_{k=0}^{\nu-1}\prod_{j=1}^{m}\mathcal{D}^{s_j}_{-|\alpha_j q^k|}\Big(|q|^{-(l-\nu)(n+\varepsilon_0)+(S-\varepsilon_0/d)(l-\nu)^2/2+a(n+\varepsilon_0)+b}\Big) = \\
&= |q|^{-\nu n-(l-\nu)(n+\varepsilon_0)+(S-\varepsilon_0/d)(l-\nu)^2/2+a(n+\varepsilon_0)+b}\prod_{k=0}^{\nu-1}\prod_{j=1}^{m}\big(1+|\alpha_j q^{k+l-\nu-a}|\big)^{s_j} \leqslant \\
&\leqslant |q|^{-ln+(S-\varepsilon_0/d)l^2/2-(1-\nu/d)\varepsilon_0 l+a(n+\varepsilon_0)+b+c_1} \leqslant |q|^{-ln+(S-\varepsilon_0/d)l^2/2+a(n+\varepsilon_0)+b+c_1},
\end{aligned} \tag{2.11}$$

where c_1 is a constant depending only on q, P, m, α_j, s_j.

The lemma follows from (2.10) and (2.11). □

Lemma 2. *Let $\vec{\omega} = (\omega_0, \omega_{j,k,\sigma}) \in \mathbb{C}^{1+dS}$ be such that*

$$\omega_0 + \sum_{j=1}^{m}\sum_{k=0}^{d-1}\sum_{\sigma=0}^{s_j-1}\omega_{j,k,\sigma}f^{(\sigma)}(\alpha_j q^k) = 0.$$

Then for $l \geqslant 0$ and $n \geqslant Sl$ we have

$$|v_{l,n}(\vec{\omega})| \leqslant \max_{j,k,\sigma}|\omega_{j,k,\sigma}| \cdot |q|^{-ln+(S-\varepsilon_0/d)l^2/2+c(n+1)},$$

where c is a positive constant depending only on q, P, m, α_j, s_j.

Proof. In the proof we denote by c_1, c_2, c_3 positive constants depending only on q, P, m, α_j, s_j.

It follows from (2.1) that

$$\omega_0 + \sum_{n=0}^{\infty} \frac{u_n(\vec{\omega})}{\prod_{k=1}^{n} P(q^k)} = \omega_0 + \sum_{j=1}^{m} \sum_{k=0}^{d-1} \sum_{\sigma=0}^{s_j-1} \omega_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) = 0.$$

Hence (2.2) gives

$$v_n(\vec{\omega}) = -\sum_{l=n+1}^{\infty} \frac{u_l(\vec{\omega})}{\prod_{k=n+1}^{l} P(q^k)}. \tag{2.12}$$

It follows from (2.1) that for $n \geqslant 1$ we have

$$|u_n(\vec{\omega})| \leqslant c_1^n \max_{j,k,\sigma} |\omega_{j,k,\sigma}|.$$

Hence (2.12) gives

$$|v_n(\vec{\omega})| \leqslant \max_{j,k,\sigma} |\omega_{j,k,\sigma}| \cdot \sum_{l=n+1}^{\infty} \frac{c_2 c_1^l}{(2c_1)^{l-n}} = c_2 c_1^n \max_{j,k,\sigma} |\omega_{j,k,\sigma}|.$$

Consequently for $0 \leqslant \nu < d$ and $n \geqslant S\nu$ we have

$$|v_{\nu,n}(\vec{\omega})| \leqslant \max_{j,k,\sigma} |\omega_{j,k,\sigma}| \cdot |q|^{c_3(n+1)} \leqslant \max_{j,k,\sigma} |\omega_{j,k,\sigma}| \cdot |q|^{-\nu n + (S-\varepsilon_0/d)\nu^2/2 + (c_3+d)n + c_3}.$$

It follows from Lemma 1 that for $l \geqslant 0$ and $n \geqslant Sl$ we have

$$|v_{l,n}(\vec{\omega})| \leqslant \max_{j,k,\sigma} |\omega_{j,k,\sigma}| \cdot |q|^{-ln + (S-\varepsilon_0/d)l^2/2 + (c_3+d)n + c_3 + (c_3+d+c')l},$$

where c' is the constant of Lemma 1. Using $l \leqslant n/S$, we obtain the lemma. $\square$

3 Non-vanishing lemma

Lemma 3. *Let $\vec{\omega} = (\omega_0, \omega_{j,k,\sigma}) \in \mathbb{C}^{1+dS}$ be such that for some $l_0, n_0 \in \mathbb{Z}_{\geqslant 0}$ with $n_0 \geqslant Sl_0$ we have*

$$v_{l_0,n_0}(\vec{\omega}) = v_{l_0,n_0+1}(\vec{\omega}) = \ldots = v_{l_0,n_0+dS}(\vec{\omega}) = 0. \tag{3.1}$$

Then the generating function $F(z)$ of the sequence $v_n(\vec{\omega})$,

$$F(z) = \sum_{n=0}^{\infty} v_n(\vec{\omega}) z^n \in \mathbb{C}[[z]],$$

is rational.

Proof. Consider the sequence $\{w_n\}_{n \geqslant 0}$ given by

$$w_n = v_{n_0 - Sl_0 + n}(\vec{\omega}) \qquad (0 \leqslant n < Sl_0),$$
$$\prod_{k=1}^{l_0} \prod_{j=1}^{m} \mathcal{D}_{\alpha_j q^{-k}}^{s_j}(w_n) = 0 \qquad (n \geqslant Sl_0),$$

where $\mathcal{D}_a$ is given by (2.4). From (2.7) and (3.1) it follows that

$$w_n = v_{n_0 - Sl_0 + n}(\vec{\omega}) \qquad (0 \leqslant n \leqslant S(l_0 + d)). \tag{3.2}$$

It follows from (2.6) that for $\nu \in \mathbb{Z}$ we have

$$\prod_{k=1}^{l_0}\prod_{j=1}^{m}\mathcal{D}^{s_j}_{\alpha_j q^{\nu-k}}\big(q^{\nu n}w_n\big) = q^{\nu n}\prod_{k=1}^{l_0}\prod_{j=1}^{m}\mathcal{D}^{s_j}_{\alpha_j q^{-k}}(w_n) = 0 \qquad (n \geqslant Sl_0).$$

Hence the sequence

$$z_n = w_{n+1} - P(q^{n_0-Sl_0+n+1})w_n - u_{n_0-Sl_0+n+1}(\vec{\omega}) \qquad (n \geqslant 0)$$

satisfies the linear recurrence relation

$$\prod_{k=-l_0}^{d-1}\prod_{j=1}^{m}\mathcal{D}^{s_j}_{\alpha_j q^{k}}(z_n) = 0 \qquad (n \geqslant S(l_0+d))$$

of order $S(l_0+d)$.

On the other hand, it follows from (2.3) and (3.2) that $z_n = 0$ for $0 \leqslant n < S(l_0+d)$. Hence $w_n = v_{n_0-Sl_0+n}(\vec{\omega})$ for all $n \geqslant 0$, i. e., $v_n(\vec{\omega})$ is linear recurrent and

$$F(z) = \sum_{n\geqslant 0} v_n(\vec{\omega})z^n \in \mathbb{C}(z).$$

This completes the proof. □

Lemma 4. *Let $\alpha_1,\dots,\alpha_m$ satisfy the conditions 1–2 of Theorem 1, $\vec{\omega} = (\omega_0, \omega_{j,k,\sigma}) \in \mathbb{C}^{1+dS}\setminus\{\vec{0}\}$. Then the generating function $F(z)$ of the sequence $v_n(\vec{\omega})$,*

$$F(z) = \sum_{n=0}^{\infty} v_n(\vec{\omega})z^n \in \mathbb{C}[[z]],$$

is not rational.

Proof. Assume the converse. Then for some constant $C > 1$ we have $|v_n(\vec{\omega})| = O(C^n)$. It follows from (2.1) and (2.2) that

$$\omega_0 + \sum_{j=1}^{m}\sum_{k=0}^{d-1}\sum_{\sigma=0}^{s_j-1}\omega_{j,k,\sigma}f^{(\sigma)}(\alpha_j q^k) = \omega_0 + \sum_{n=0}^{\infty}\frac{u_n(\vec{\omega})}{\prod_{k=1}^{n}P(q^k)} = 0.$$

In particular, not all $\omega_{j,k,\sigma}$ vanish.

From (2.3) it follows that $F(z)$ satisfies the functional equation

$$(1-p_0 z)F(z) = \sum_{\nu=1}^{d} p_\nu q^\nu z F(q^\nu z) + R(z), \tag{3.3}$$

where

$$P(z) = \sum_{\nu=0}^{d} p_\nu z^\nu,$$

$$R(z) = \omega_0 + \sum_{n=0}^{\infty} u_n(\vec{\omega})z^n = \omega_0 + \sum_{j=1}^{m}\sum_{k=0}^{d-1}\sum_{\sigma=0}^{s_j-1}\frac{\omega_{j,k,\sigma}\sigma!z^\sigma}{(1-\alpha_j q^k z)^{\sigma+1}} \in \mathbb{C}(z).$$

The condition 1 of Theorem 1 implies that all $\alpha_j q^k$ are different. Since not all $\omega_{j,k,\sigma}$ vanish, the function $R(z)$ has at least one pole. It follows from (3.3) that $F(z)$ also has a pole in $\mathbb{C}^*$.

We claim that any pole of $F(z)$ is of the form $\alpha_j^{-1}q^n$ with $n \in \mathbb{Z}_{>0}$. Assume the contrary. Let β be a pole that cannot be represented in this form with the least $|\beta|$. Then $R(z)$ doesn't have a pole at the point βq^{-d}. It follows from (3.3) that one of the functions $F(q^\nu z)$ with $0 \leqslant \nu < d$ has a pole at βq^{-d}. Hence we have $\beta = \beta' q^{d-\nu}$ for some pole β' of $F(z)$. But then $|\beta'| < |\beta|$. Consequently β' can be represented in the required form as well as β. This contradiction proves our claim about poles of $F(z)$. In particular, it follows from the condition 1 of Theorem 1 that $F(z)$ and $R(z)$ do not have common poles.

Now suppose β is a pole of $F(z)$ with maximal $|\beta|$. It follows from (3.3) and the above that the function $(1 - p_0 z)F(z)$ does not have a singularity at the point β. Hence $p_0\beta = 1$. Since $\beta = \alpha_j^{-1}q^n$ with $n \in \mathbb{Z}_{>0}$, this contradicts the condition 2 of Theorem 1. This contradiction proves the lemma. $\square$

From Lemmas 3 and 4, we get the following non-vanishing lemma.

Lemma 5. *Let $\alpha_1, \ldots, \alpha_m$ satisfy the conditions 1–2 of Theorem 1, $\vec{\omega} = (\omega_0, \omega_{j,k,\sigma}) \in \mathbb{C}^{1+dS} \setminus \{\vec{0}\}$. Then for any $l_0, n_0 \in \mathbb{Z}_{\geqslant 0}$ with $n_0 \geqslant Sl_0$ there exists an integer n with $n_0 \leqslant n \leqslant n_0 + dS$ such that $v_{l_0,n}(\vec{\omega}) \neq 0$.* $\square$

4 Main proposition

Suppose $\alpha_j \in \mathbb{Q}^*$ $(1 \leqslant j \leqslant m)$. Denote by D any positive integer such that $DP(z) \in \mathbb{Z}[z]$ and $D\alpha_j q^k \in \mathbb{Z}$ for $1 \leqslant j \leqslant m$, $0 \leqslant k < d$. For $l, n \in \mathbb{Z}_{\geqslant 0}$ with $n \geqslant Sl$ consider

$$w_{l,n} = w_{l,n}(\vec{x}) = D^n q_1^{Sl(l+1)/2} q_2^{dn(n+1)/2} v_{l,n}(\vec{x}).$$

It follows from (2.1) and (2.2) that

$$D^n q_2^{dn(n+1)/2} v_n \in \mathbb{Z}[\vec{x}] \qquad (n \geqslant 0).$$

Combining this with (2.7), we get $w_{l,n} \in \mathbb{Z}[\vec{x}]$.

For a linear form L denote by $\mathcal{H}(L)$ the maximum of absolute values of its coefficients. From (2.1) and (2.2) it follows that

$$\mathcal{H}(v_n) \leqslant |q|^{dn^2/2+O(n+1)}.$$

In view of (2.7) the same estimate is valid for $\mathcal{H}(v_{l,n})$ $(n \geqslant Sl \geqslant 0)$. Finally, for $w_{l,n}$ we have

$$\mathcal{H}(w_{l,n}) \leqslant |q_1|^{dn^2/2+Sl^2/2+O(n+1)} \qquad (n \geqslant Sl \geqslant 0).$$

The above can be summarized as follows.

Proposition 1. *Under the hypotheses of Theorem 1, for any $l, n \in \mathbb{Z}_{\geqslant 0}$ with $n \geqslant Sl$ there exists a linear form $w_{l,n} = w_{l,n}(\vec{x}) \in \mathbb{Z}[\vec{x}]$ such that the following conditions hold:*

1. $\mathcal{H}(w_{l,n}) \leqslant |q_1|^{dn^2/2+Sl^2/2+O(n+1)}$,

2. *for any $\vec{\omega} = (\omega_0, \omega_{j,k,\sigma}) \in \mathbb{C}^{1+dS}$ such that*

$$\omega_0 + \sum_{j=1}^{m} \sum_{k=0}^{d-1} \sum_{\sigma=0}^{s_j-1} \omega_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) = 0$$

we have

$$|w_{l,n}(\vec{\omega})| \leqslant \max_{j,k,\sigma} |\omega_{j,k,\sigma}| \cdot |q_1|^{\gamma dn^2/2-(1-\gamma)ln+\left((1-\gamma/2)S-(1-\gamma)\varepsilon_0/(2d)\right)l^2+O(n+1)},$$

where γ and ε_0 are given by (1.1) and (2.8),

3. *for any $\vec{\omega} = (\omega_0, \omega_{j,k,\sigma}) \in \mathbb{C}^{1+dS} \setminus \{\vec{0}\}$ and $l_0, n_0 \in \mathbb{Z}_{\geqslant 0}$ with $n_0 \geqslant Sl_0$ there exists an integer n with $n_0 \leqslant n \leqslant n_0 + dS$ such that $w_{l_0,n}(\vec{\omega}) \neq 0$.*

The constants in the Landau symbols $O(\cdot)$ depend only on q, P, m, α_j, s_j. □

5 Proof of Theorem 1

Take

$$n_0 = \left\lceil \frac{dS - \varepsilon_0/2 + \sqrt{(dS)^2 + (1-\varepsilon_0)dS + \varepsilon_0^2/4}}{d}\, l \right\rceil = \left\lceil \frac{(M-1)l}{d} \right\rceil \geqslant Sl,$$

where M is given by (1.3) and $l \in \mathbb{Z}_{\geqslant 0}$ will be chosen later. It follows from Proposition 1 that there exists an integer $n = n_0 + O(1)$ such that $w_{l,n}(\vec{A}) \neq 0$. Since $w_{l,n} \in \mathbb{Z}[\vec{x}]$, we get

$$|w_{l,n}(\vec{A})| \geqslant 1.$$

Let $\vec{\omega} = (\omega_0, \omega_{j,k,\sigma})$ be given by

$$\omega_{j,k,\sigma} = A_{j,k,\sigma},$$
$$\omega_0 = -\sum_{j=1}^{m}\sum_{k=0}^{d-1}\sum_{\sigma=0}^{s_j-1} \omega_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k).$$

Using Proposition 1, we get

$$|w_{l,n}(\vec{\omega})| \leqslant H|q_1|^{-al^2+O(l+1)},$$

where

$$a = \frac{1-M\gamma}{d}\sqrt{(dS)^2 + (1-\varepsilon_0)dS + \varepsilon_0^2/4}.$$

Take $l = (L/a)^{1/2} + O(1)$, where $L = \frac{\log H}{\log |q_1|}$, such that

$$|w_{l,n}(\vec{\omega})| \leqslant 1/2.$$

Then we have

$$|w_{l,n}(\vec{A}) - w_{l,n}(\vec{\omega})| \geqslant 1/2.$$

On the other hand, using Proposition 1, we get

$$|w_{l,n}(\vec{A}) - w_{l,n}(\vec{\omega})| \leqslant \mathcal{H}(w_{l,n})|A_0 - \omega_0| \leqslant |A_0 - \omega_0| \cdot |q_1|^{\mu L + O(L^{1/2})},$$

where μ is given by (1.4). Since

$$|A_0 - \omega_0| = \left| A_0 + \sum_{j=1}^{m}\sum_{k=0}^{d-1}\sum_{\sigma=0}^{s_j-1} A_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) \right|,$$

we obtain Theorem 1.

References

[1] M. Amou, K. Väänänen, "Linear independence of the values of q-hypergeometric series and related functions", *Ramanujan J.* **9**:3 (2005), 317–339; doi 10.1007/s11139-005-1871-8, MR 2173492, Zbl 1095.11038.

[2] M. Katsurada, "Linear independence measures for values of Heine series", *Math. Ann.* **284**:3 (1989), 449–460; doi 10.1007/BF01442496, MR 1001713, Zbl 0653.10031.

[3] I. Rochev, "Linear independence of values of certain q-series" (in Russian), accepted for publication in Izv. RAN. Ser. Mat.

[4] O. Sankilampi, K. Väänänen, "On the values of Heine series at algebraic points", *Results Math.* **50**:1–2 (2007), 141–153; doi 10.1007/s00025-006-0240-2, MR 2313136.